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Models of Relativistic Hamiltonian Interactions

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Abstract

A model of a relativistically invariant Hamiltonian 2-particle interaction is given. It is classical in the sense of having 6 degrees of freedom. This model shows that the author's earlier general definition of such systems is not vacuous. In this model the forces of interaction die away as the particles are removed from each other.

1. *Definitions*

The general method for constructing such Lorentz-invariant systems of interacting particles was presented in (Arens, 1974). These systems are completely Hamiltonian (see footnote 1). Briefly, the entire class of interactions defined in (Arens, 1974) is obtained in the following way.

Say there are two particles. Let M be four-dimensional Cartesian space $\mathbb{R}^4$, regarded as space-time. Form $M \times M$ with coordinates $x^1, x^2, x^3, x^4, y^1, y^2, y^3, y^4$. Then form $T_1(M \times M)$, the cotangent bundle over $M \times M$ with coordinates $x^1, x^2, \ldots, y^4, p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4$. In $T_1(M \times M)$ there is defined a symplectic structure and a Poisson bracket

$$\{F, G\} = \frac{\partial F}{\partial x^i}\frac{\partial G}{\partial p_i} + \frac{\partial F}{\partial y^i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial x^i} - \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial y^i}$$

where, as henceforth, we sum on repeated indices from 1 to 4.

We now select a surface $\mathcal{S}_2$ in $T_1(M \times M)$ invariant under the standard action of the Poincaré group, having also properties H-1, H-2, H-3 as follows (footnote 2):

[1] As emphasized in (Arens, 1974), these systems do not have all the properties enumerated in (Currie, Jordan, and Sudarshar, 1963).

[2] We present them in a different order from, but with the same numbering as that given in (Arens, 1974). The wording in (Arens, 1974) is more complicated because invariance is not there postulated at the outset.

(H-2). There must exist functions H_1, H_2 depending only on x^1, x^2, x^3, x^4, $y^1, y^2, y^3, y^4, p_1, p_2, p_3, q_1, q_2, q_3$, but defined for all values of these variables satisfying

$$(x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2 > (x^4 - y^4)^2 \tag{1.1}$$

such that $\mathscr{S}_2$ is described by

$$p_4 + H_1 = 0, q_4 + H_2 = 0 \tag{1.2}$$

H_1 and H_2 are subject to the condition

(H-1)
$$\{p_4 + H_1, q_4 + H_2\} = 0 \tag{1.3}$$

A *motion* is a maximal connected integral submanifold for the singular distribution of the restriction to S_2 of the symplectic structure of $T_1(M \times M)$ (Arens, 1974).

These motions are 2-dimensional.

(H-3). Each pair of values (x^4, y^4) occurs once and only once on each motion.

If these conditions are formulated for only one particle instead of two then (H-1) becomes vacuous, (H-2) says that there is a Hamiltonian while (H-3) says that given any initial conditions there is a motion existing for all times x^4.

Returning to two particles, it is shown in (Arens, 1974) that the resulting system has a Hamiltonian

$$\begin{aligned} H(x^1, x^2, x^3, y^1, y^2, y^3, p_1, p_2, p_3, q_1, q_2, q_3, t) \\ = H_1(x^1, x^2, x^3, t, y^1, y^2, y^3, t, p_1, p_2, p_3, q_1, q_2, q_3) \\ + H_2(x^1, x^2, x^3, t, y^1, y^2, y^3, t, p_1, p_2, p_3, q_1, q_2, q_3) \end{aligned} \tag{1.4}$$

The system will have zero interaction if and only if the six functions $\{\partial H/\partial x^i, H\}\{\partial H/\partial y^i, H\}$ are identically zero. It was argued in Arens (1974) that there is no *formal* obstacle to having a nonzero interaction. In the present paper we want to exhibit examples to show that (H-2) and (H-1) can be satisfied without having a zero interaction. Reference to our remarks about single particles shows that (H-3) is a still more difficult matter. Since we do not exhibit our H_1 and H_2 explicitly, it is difficult to establish the required Lipschitz conditions for our examples. At any rate, we have not established (H-3) for our examples.

2. *Defining an Invariant* S_2

We will insure the invariance of $\mathscr{S}_2$ under the Poincaré group by defining it by equations

$$F_1 = 0, F_2 = 0 \tag{2.1}$$

where F_1 and F_2 are functions on $T_1(M \times M)$ which are invariant under the group. (We are then still permitted to discard components of the surface defined by 2.1.)

For $a = (a^1, a^2, a^3, a^4)$ and $b = (b^1, b^2, b^3, b^4)$ we define $a \cdot b = -a^1b^1 - a^2b^2 - a^3b^3 + a^4b^4$. We let $x = (x^1, x^2, x^3, x^4)$, etc. We let $z = x - y$. We let

$$\rho = p \cdot p, \nu = p \cdot q, \sigma = q \cdot q, \lambda = p \cdot z, \zeta = z \cdot z, \mu = -q \cdot z \tag{2.2}$$

If F_1, F_2 depend only on these 6 functions, the invariance will be insured.

3. *The Condition* (H-1)

If F_1 and F_2 satisfy $\{F_1, F_2\} = 0$, and if the jacobian $\partial(F_1, F_2)/\partial(p_4, q_4)$ is not 0, we can solve the equations *locally* for p_4, q_4: $p_4 = -H_1, q_4 = -H_2$ and (H-1) will result locally. Our problem is thus two-fold: (a) ensure that $\{F_1, F_2\} = 0$ and (b) make sure that these equations (2.1) have a global solution. We consider the first problem in this section.

We will, as a matter of fact, deal only with the choice of $F_1 = \rho - \sigma$. (This certainly includes the case of zero interaction, which is given by $F_2 = \rho - 1$. This gives two free particles.) The following is obviously relevant.

Theorem. Let F be an invariant function on $T_1(M \times M)$. Let F be expressed in terms of $\rho, \nu, \sigma, \lambda, \zeta, \mu$. Then $\{\rho - \sigma, F\} = 0$ if and only if F depends only on $\rho, \sigma, \nu, \Delta$, or Γ where Δ is the Gram matrix

$$\det \begin{pmatrix} (p+q)\cdot(p+q) & (p+q)\cdot z \\ (p+q)\cdot z & z \cdot z \end{pmatrix} \tag{3.1}$$

and Γ is the Gram matrix

$$\det \begin{pmatrix} p \cdot p & p \cdot q & p \cdot z \\ p \cdot q & q \cdot q & q \cdot z \\ p \cdot z & q \cdot z & z \cdot z \end{pmatrix} \tag{3.2}$$

Proof: We recall the Poisson bracket table for the six invariants [Arens, 1974 (5.1)]

$$\begin{array}{cccccc} 0 & 0 & 0 & 2\rho & 4\lambda & -2\nu \\ 0 & 0 & 0 & \nu - \rho & -2(\lambda + \mu) & \nu - \sigma \\ 0 & 0 & 0 & -2\nu & 4\mu & 2\sigma \\ -2\rho & \rho - \nu & 2\nu & 0 & 2\zeta & \mu - \lambda \\ -4\lambda & 2(\lambda + \mu) & -4\mu & -2\zeta & 0 & -2\zeta \\ 2\nu & \sigma - \nu & -2\sigma & \lambda - \mu & 2\zeta & 0 \end{array} \tag{3.3}$$

From this it follows at once that $\{\rho - \sigma, F\} = 0$ if and only if

$$(\rho + \nu)\frac{\partial F}{\partial \lambda} + 2(\lambda - \mu)\frac{\partial F}{\partial \zeta} - (\sigma + \nu)\frac{\partial F}{\partial \mu} = 0 \tag{3.4}$$

It is routinely verifiable that the five given functions satisfy this condition. But there can be at most five functionally independent functions that do this or every invariant function would commute with $\rho - \sigma$. But λ, for example, does not. This proves the present theorem.

Actually, we noted earlier that the two functions

$$\chi \equiv (p+q)\cdot(p+q) = \rho + 2\nu + \sigma \tag{3.5}$$

and Γ commute with everything. An interesting example of a function commuting with $\rho - \sigma$ is

$$\psi = \frac{\lambda}{\nu+\rho} + \frac{\mu}{\nu+\sigma} \tag{3.6}$$

One has the relation

$$(\nu+\rho)^2(\nu+\sigma)^2\psi^2 + \chi\Gamma + (\nu^2 - \rho\sigma)\Delta = 0 \tag{3.7}$$

With $F_1 = \rho - \sigma$ and F_2 any function of the five functions in the theorem, we will insure invariance and also $\{F_1, F_2\} = 0$. The choice of F_2 is limited by these conditions: we want to be able to solve $F_1 = 0, F_2 = 0$ for p_4 and q_4 and we want to avoid getting a zero interaction. (A zero interaction would surely result if F_2 depended only on ρ, σ, and ν.)

In order to explore this we introduce coordinates as follows.

$$\begin{aligned} p &= (p_1, p_2, p_3, u), \qquad & P &= (p_1, p_2, p_3) \\ q &= (q_1, q_2, q_3, v), \qquad & Q &= (q_1, q_2, q_3) \\ z &= (z_1, z_2, z_3, t), \qquad & Z &= (z_1, z_2, z_3) \end{aligned} \tag{3.8}$$

We abbreviate $p_1q_1 + p_2q_2 + p_3q_3$ by $P\cdot Q$, etc. We write P^2 for $P\cdot P$ and $|P|$ for $(P\cdot P)^{1/2}$, etc. We also abbreviate $(P+Q)\cdot Z$ by a, and $|P+Q|$ by b.

We will confine ourselves to that region of $T_1(M\times M)$ where $t^2 < Z^2$, $u > 0, v > 0$. What that means is that if we define $\mathscr{S}_2$ by equations $F_1 = 0$, $F_2 = 0$ then we count as belonging to $\mathscr{S}_2$ only those points of $T_1(M\times M)$ for which also $t^2 < Z^2$ and $u > 0, v > 0$. We could just as well have used the relations $u < 0, v < 0$, although this might seem a little odd and unmotivated at this point. The fact is, the two choices ultimately result in Hamiltonians of opposite signs (see Section 5).

Our intent is now to express our 5 functions in terms of u, v, t and the Euclidean invariants of P, Q, Z. For example, $\rho = u^2 - P^2$, $\sigma = v^2 - Q^2$. It is a remarkable fact that we can replace u and v by $\delta \equiv \rho - \sigma$ and $w = u + v$. Our equations are then $\delta = 0, F_2 = 0$ and we will have to solve the latter for w. Having done so we get u and v from $u^2 = \rho + P^2$, $v^2 = \sigma + Q^2$.

$$\begin{aligned} \chi = \rho + 2\nu + \sigma &= u^2 - P^2 + 2(uv - P\cdot Q) + v^2 - Q^2 \\ &= w^2 - (P+Q)^2 = w^2 - b^2 \end{aligned}$$

For Δ we need $\lambda - \mu$. $\lambda - \mu = (p+q)\cdot z = wt - a$. So

$$\Delta = \chi\zeta - (\lambda-\mu)^2 = (w^2 - b^2)(t^2 - Z^2) - (wt-a)^2$$
$$= -Z^2w^2 + 2atw - a^2 + b^2(Z^2 - t^2) \tag{3.9}$$

Finally, we compute ρ and σ.
We note

$$w^2 = u^2 + 2uv + v^2$$
$$= \rho + \sigma - P^2 - Q^2 + 2uv \tag{3.10}$$

On the other hand

$$\frac{1}{w^2} = \frac{1}{u^2+v^2+2uv} = \frac{u^2+v^2-2uv}{(u^2-v^2)^2} \tag{3.11}$$

Now $u^2 - v^2 = \delta + P^2 - Q^2$ where $\delta = \rho - \sigma$. Thus

$$u^2 + v^2 - 2uv = \frac{(\delta + P^2 - Q^2)^2}{w^2} \tag{3.12}$$

Hence

$$w^2 + \frac{(\delta+P^2-Q^2)^2}{w^2} = 2(u^2+v^2) = 2(\rho+\sigma+P^2+Q^2) \tag{3.13}$$

We add and subtract $2\delta = 2(\rho - \sigma)$ and obtain

$$\rho = \frac{1}{4}\left[w^2 - 2(P^2+Q^2-\delta) + \frac{(P^2-Q^2+\delta)^2}{w^2}\right]$$
$$\sigma = \frac{1}{4}\left[w^2 - 2(P^2+Q^2+\delta) + \frac{(P^2-Q^2+\delta)^2}{w^2}\right] \tag{3.14}$$

4. *Solving for w*

As we said, we will make $\delta = 0$. Therefore in this section ρ will stand for

$$\frac{1}{4}\left[w^2 - 2(P^2+Q^2) + \frac{(P^2-Q^2)^2}{w^2}\right] \tag{4.1}$$

The strategy is to assemble a list of functions of w which are monotonely increasing with w for all values of $t, a, b, \ldots$ and whose range of values covers some fixed open interval for all values of those parameters. Let F be such a function and let c belong to its range. Then $F = c$ can be solved for w in terms of $t, a, b, \ldots$

The function χ is an obvious example. It is monotone on $(0, \infty)$ and its range includes all positive values. However, it does not involve Z and so would produce a zero interaction. The function that makes examples possible is described in the following.

Lemma. Let $n \geq 2$ and denote by $f(w)$ the quotient

$$\frac{\Delta}{\chi^n} \equiv \frac{-Z^2w^2 + 2atw - a^2 + b^2(Z^2 - t^2)}{(w^2 - b^2)^n} \tag{4.2}$$

for $w \geq b$. Then

$$f'(w) \geq -Z^2 \frac{\mathrm{d}}{\mathrm{d}w}\left[\frac{(w-b)^2}{(w^2-b^2)^n}\right]$$

$$-\frac{Z^2}{(w^2-b^2)^{n-1}} \leq f(w) \leq -\frac{Z^2}{(w+b)^2(w^2-b^2)^{n-2}} \tag{4.3}$$

and $f(w)$ increases from $-\infty$ to 0 as w varies from b to ∞.

Proof: First consider the case $b \neq 0$. Let $|Z| = (Z \cdot Z)^{1/2}$.
Let $w = bs, a = \alpha b|Z|, t = \beta|Z|$. Then

$$f(w) = -Z^2 b^{2-2n} g(s) \tag{4.4}$$

where

$$g(s) = \frac{s^2 - 2\alpha\beta s - 1 + \alpha^2 + \beta^2}{(s^n - 1)^n} \tag{4.5}$$

Our next step is to show that $g'(s)$ attains its maximum (for fixed s), when $\alpha = \beta = 1$. Examination of $g'(s)$ shows that it attains its maximum whenever $\alpha\beta(1 - s^2 + 2ns^2) - ns(\alpha^2 + \beta^2)$ attains its maximum. Evidently, this maximum occurs with $\alpha\beta > 0$. The value for $\alpha = \beta = 1$ is positive, hence the maximum is positive, so that we conclude that it occurs on the boundary where either α or β is 1. For $\beta = 1$ we have $\alpha(2ns^2 - s^2 + 1) - ns(\alpha^2 + 1)$. The critical point for α is greater than 1, but the maximum for $0 \leqq \alpha \leqq 1$ is at 1. Going all the way back to f, and observing the sign change, we see that

$$f'(w) \geq \text{the value of } f'(w) \text{ when } a = b|Z|, t = |Z|$$

$$\geq \frac{d}{dw}\left[-Z^2 \frac{(w-b)^2}{(w^2-b^2)^n}\right] \tag{4.6}$$

This is one thing we promised to show. Examination of the derivative on the right shows that $f'(w) > 0$ for $b < w < \infty$, as we asserted.

We also see that

$$f(w) + \frac{Z^2(w-b)^2}{(w^2-b^2)^n} \tag{4.7}$$

never decreases, so it is not greater than its limit at ∞. Hence f is bounded above by the expression claimed. For the lower bound we observe that

$$g(s) = \frac{(s-\alpha\beta)^2 - (1-\alpha^2)(1-\beta^2)}{(s^2-1)^n} \leq \frac{(s-\alpha\beta)^2}{(s^2-1)^n} \leqq \frac{(s-1)^2}{(s^2-1)^n} \tag{4.8}$$

To finish the proof for $b \neq 0$ we must examine the value $f(b)$. This is obviously 0.

Now when $b = 0$ we have $f(w) = -Z^2 w^{2-2n}$, and everything the lemma says obviously holds in this case also.

Corollary. As w varies from b to $+\infty$, the function

$$\left(1 - \frac{\chi^n}{\Delta}\right)^{-1} \tag{4.9}$$

is positive, and is strictly decreasing, if $n \geq 2$.

We now state some obvious facts about ρ.

For $w \geq |P| + |Q|$, ρ is monotonely increasing to ∞.

At $w = |P| + |Q|$, ρ is zero

$$|P| + |Q| \geq b \tag{4.10}$$

As a result of the corollary and (4.10), taking $n > 2$, we can let our second equation be

$$[1 - (\chi^n/\Delta)]^{-1} - \rho = 0 \tag{4.11}$$

because the graphs of ρ and

$$[1 - (\chi^n/\Delta)]^{-1} \tag{4.12}$$

certainly do cross somewhere in the upper half-plane where ρ is monotonely increasing. Thus the value of w is greater than $|P| + |Q|$. The corresponding value of ρ (and σ) is then positive, and $u = (\rho + P^2)^{\frac{1}{2}}$, $v = (\sigma + Q)^{\frac{1}{2}}$ are defined.

The reader should check that all this is true even for $b = 0$. Of course we want to be sure that w depends differentiably on the parameters $t, a, b, \ldots$ This can be shown by an easy application of the implicit function theorem.

Another equation which works for the same reasons (always understanding $n \geq 2$) is

$$-\frac{\Delta}{\chi^n} - 1 = \rho - 4\chi \tag{4.13}$$

This one works because

$$\rho - 4\chi = \frac{(P^2 - Q^2)^2}{w^2} - (P - Q)^2 \tag{4.14}$$

which has obvious monotonicity and range properties.

In Section 7 we point out why this equation is not as good as (4.11).

The reader may have wondered why we did not use the equation

$$\Delta\chi^{-n} = -1 \tag{4.15}$$

This is not satisfactory for a reason to be discussed in the next section. Even more undesirable would be

$$\chi = 1 \tag{4.16}$$

5. *The Hamiltonian*

According to the general theory [Arens, 1974 (4.4)] the Hamiltonian of the system is obtained as follows. Write $\mathscr{S}_2$ as the locus of $p_4 + H_1 = 0$, $q_4 + H_2 = 0$. Then

$$H = (H_1 + H_2)|_{x^4 = y^4 = 0} \tag{5.1}$$

Obviously this says that H is nothing but $-w$ with t set equal to 0. Specifically, for (4.11) say with $n = 2$, the Hamiltonian satisfies

$$\left(1 - \frac{(H^2 - b^2)^2}{-Z^2H^2 - a^2 + b^2Z^2}\right)^{-1} = \frac{1}{4}\left[H^2 - 2(P^2 + Q^2) + \frac{(P^2 - Q^2)^2}{H^2}\right] \tag{5.2}$$

Our theorems on monotonicity guarantee that this has exactly one *negative* solution. (If a negative Hamiltonian is undesirable, it can be avoided by having $\mathscr{S}_2$ lie in the region $u < 0, v < 0$.)

One can let $H = -S^{1/2}$. Then S satisfies what is equivalent to a polynomial equation of the fourth degree:

$$\left[S - 2(P^2 + Q^2) + \frac{(P^2 - Q^2)^2}{S}\right]\left[1 + \frac{(S - b^2)^2}{(S - b^2)Z^2 + a^2}\right] = 4 \tag{5.3}$$

We have shown that this has always a unique positive root S. We remind the reader that $a = (P + Q) \cdot Z$ and $b = |P + Q|$.

A Hamiltonian is supposed to satisfy the *Hessian condition*, which is to say that the (Hessian) matrix

$$\left(\frac{\partial^2 H}{\partial p_i, \partial p_j}\right)_{i,j=1,\ldots,n} \tag{5.4}$$

is nonsingular. Here the $p_1, \ldots, p_n$ are the momenta in some coordinate system. It should be possible to decide whether this Hessian condition holds for an H defined by (5.3), but it seems somewhat premature to study this problem when there is no good physical justification for this H. However, we can prove rather easily that the Hessian determinant for H given by (5.3) is not identically zero.

To see this observe first that H is an algebraic function of Z^{-2} which is analytic for $Z^{-2} = 0$. Now when Z^{-2} is 0 we get

$$H = -(1 + P^2)^{\frac{1}{2}} - (1 + Q^2)^{\frac{1}{2}} \tag{5.5}$$

This is a familiar Hamiltonian for two noninteracting particles of mass 1 and it is wellknown, and easily seen, to have a nonsingular Hessian.

It is rather easy to see that for $Z^{-2} \neq 0$, the H does not depend only on the momenta, and hence we really have nonzero interaction.

The Hessian condition is violated when H is based on (4.15) or (4.16) because H then depends only on a and b, and their dependence on the momenta is only *via* the three components of $P + Q$. Thus $\partial H/\partial p_1 = \partial H/\partial q_i$ (etc.) and so the Hessian determinant is identically zero.

To verify the Hessian condition for (4.13) completely is also rather hard. Formally, we can set $Z = 0$. This is physically unrealistic, and the Hamiltonian has (when $Z = 0$) a singularity when $(P - Q)^2 \leqq 1$. But the Hessian determinant is not 0 when $(P - Q)^2 > 1$, and hence it cannot vanish identically in the physically interesting region where $Z \neq 0$.

It is possible to define formally the most general Hamiltonian resulting from the choice of $\rho - \sigma$ as F_1 (see Section 3). One simply imposes one functional relation on some four independent functions which commute with $\rho - \sigma$. In this equation one sets the interparticle time separation t equal to zero and solves for w. Then $H = -w$. To illustrate, let us take $4\rho - \chi$, χ, $-\Delta$, and $-\chi\psi/2$. With $t = 0$ and $w^2 = S$ these have the form $(P^2 - Q^2)^2/S - (P - Q)^2$, $S - b^2$, $Z^2 S + a^2 - b^2 Z^2$, and $(P - Q) \cdot Z$ respectively. Hence any relation of the form

$$(P^2 - Q^2)/S - (P - Q)^2 = f(S - b^2, Z^2 S + a^2 - b^2 Z^2, (P - Q) \cdot Z) \tag{5.6}$$

provided it can be solved for a positive root S, gives an $H = -S^{1/2}$. This must then still be examined for the Hessian condition. What we have done for (4.11) and (4.13) is to provide two simple instances of such an f.

We call this approach "formal" because to satisfy our full definition one has to be able to solve the functional relation with any t numerically less than $|Z|$. This we did for (4.11) and (4.13).

6. *Interparticle Separation*

It is natural to require of a several-particle interaction that the interaction should tend to zero for any particle removed far away from the others. For two particles this amounts to requiring that $\partial H/\partial x^i$ (and of course also $\partial H/\partial y^i$) should approach 0 as $|Z| \to \infty$.

It is not hard to see that the model based on (4.11) meets this requirement. We have already remarked that H itself has a limit (5.5) for $|Z| \to \infty$. One can now take the derivative of both sides of (5.3) with respect to x^i and again let $|Z| \to \infty$. The result of the computation together with the observation that the limit of S itself is not 0, shows that the limit of $\partial S/\partial x^i$ is 0. Again, since the limit of H is not 0, we obtain the limit for $\partial H/\partial x^i$ to be 0, as desired.

On the other hand, this requirement cannot be established for the H based on (4.13) which is why we deemed it inferior to (4.11).

7. *Symplectic Actions of the Space-Time Group*

As shown in (Arens, 1974), the concept of 2-particle system involved here can be used to construct a symplectic (or Poisson-bracket preserving) action of the Poincaré group in the (12-dimensional) phase space appropriate to a 2-particle system. On the other hand, a symplectic action of the group need not in general arise in this way from systems of our type.

Symplectic actions in $\mathbb{R}^{6N}$ as models for N-particle interactions were presented in (Thomas and Bakamjian, 1953), and studied in (Foldy, 1961) (and references to earlier work given). In these papers, the nonfulfillment of the world line condition for symplectic actions is recognized, (although they predate (Currie *et al*, 1963)). However, the question of degeneracy of the Hamiltonian is not discussed, and in fact their Hamiltonians can be degenerate.

We will show in a subsequent paper that if our *global* axion H-2 is replaced by a *local* solvability condition, H-2 (loc), one can still obtain a symplectic action. This condition H-2 (loc) is so easy to satisfy that one can generate multitudes of symplectic actions and in fact produce examples violating a condition conjectured (not claimed) to be necessary in (Thomas and Bakamjian, 1953).

The paper (Van Dam and Wigner, 1965) is about world lines, so those systems satisfy the worldline condition. Hence they cannot be (and are not claimed to be) completely Hamiltonian.

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